Another comparison of the results of Eq. (7) with centerline pressure data showing typical agreement is given in Fig. 4. Here $U_j/U_\infty = 3.3$ and r_o/r_j is taken as 0.9. In this case, Vogler's data indicate that the jet should be nearly sonic.

The pressure coefficient contours given by Vogler[6] exhibit a wake-like appearance near the $\omega = \pi$ ray. It can be seen that the ω dependence of Eq. (7) is too simple to permit such a shape. Since there is good agreement along $\omega = \pi$, it is not expected that the agreement could also be good at, say, $\omega = 3\pi/4$. One other restriction is to be placed on the model. Since there is a singularity in the velocity around the orifice lip, the pressure coefficient goes to minus infinity there. This region should be excluded from consideration.

Surprisingly, Eq. (7) predicts fair results even for the case of an underexpanded sonic jet,[9] which far exceeds the assumption used in deriving it.

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Energy Climbs, Energy Turns, and Asymptotic Expansions

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INTEREST persists in energy approximations to optimal aircraft flight in spite of various gaps and shortcomings.[1–9] The essential feature of the energy type of approximation is reduction in the order of the state differential system. This is a blessing in permitting closed form solutions, but it becomes a curse when the problem of transitions from and to specified end conditions is faced. One possible method of overcoming the difficulties is the use of asymptotic expansions for the transitions. This Note outlines this approach and extends the energy method to three-dimensional maneuvers.

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Energy Climb

With $h \equiv$ altitude, $V \equiv$ airspeed, $\gamma \equiv$ path angle to horizontal, the equations of symmetric flight are

$$\dot{h} = V \sin\gamma \tag{1}$$

$$\dot{V} = (g/W)[T(h,V) - D(h,V,\alpha)] - g \sin\gamma \tag{2}$$

$$\dot{\gamma} = (g/WV)\ [L(h,V,\alpha)] - (g/V) \cos\gamma \tag{3}$$

The thrust T is assumed to act along the flight path. The lift L and drag D are functions of altitude, airspeed, and the angle of attack α. Weight W is approximated as constant.

Equation (2) can be replaced by Eq. (4) in terms of a total energy variable E defined by Eq. (5).

$$\dot{E} = (T - D)V/W \tag{4}$$

$$E \equiv h + V^2/2g \tag{5}$$

Energy approximations can result from several different sets of assumptions. Perhaps the simplest is that drag is a function of altitude and velocity only. The lift is then decoupled from the drag and can be chosen to satisfy Eq. (3). If the lift is assumed to be unbounded, any required γ history can be followed and γ may be chosen as the control variable with h and V as the state variables. With this model, conventional variational treatment leads to a *singular arc* given by

$$(\partial/\partial h)[V(T - D)/W]_{E=\text{const}} = 0 \tag{6}$$

and to vertical climbs and dives, $\cos\gamma = 0$, as transitions from and to prescribed h, V end conditions. The large jumps in path angle are distressing in that the composite path is not smooth enough to serve as a reference trajectory for a conventional expansion and it turns out that estimated corrections for maneuvering drag effect upon speed through the "corners" diverge. The considerable value of the singular practical energy climb arcs found in applications work (Refs. 7–9) however, appears to make further effort at transition fairings worthwhile.

A seemingly grosser approximation which produces further reduction in system order will now be examined. Equations (1, 4, and 3) may be rewritten as

$$\epsilon\dot{h} = V \sin\gamma = [2g(E - h)]^{1/2} \sin\gamma \tag{7}$$

$$\dot{E} = [V(T - D)/W] = [2g(E - h)]^{1/2}\ [(T - D)/W] \tag{8}$$

$$\epsilon\dot{\gamma} = (g/V)[(L/W) - \cos\gamma] \tag{9}$$

which is the same for $\epsilon = 1$ as for the original system. We seek to approximate the solution by an expansion in ϵ around that obtained for $\epsilon = 0$. The order is reduced to one for $\epsilon = 0$, the state being E, and the control h. With this model, no transitions at all are obtained in the zero-order solution. Rather, instantaneous transitions at constant energy level[1,2,3,9] are implied, which are even more unrealistic than the instantaneous path angle changes of the two state variable model.

Energy Turns

It is of interest to examine the optimal turn problem for the possibility of an expansion around a highly simplified "zero-order" approximation similar to that for the symmetric case. Assuming coordinated turns with no side force, the following equations apply.[10]

$$\epsilon\dot{h} = V \sin\gamma \tag{10}$$

$$\dot{E} = (T - D)V/W \tag{11}$$

$$\epsilon\dot{\gamma} = (g/V)[L \cos\mu/W - \cos\gamma] \tag{12}$$

$$\dot{\chi} = gL \sin\mu/WV \cos\gamma \tag{13}$$

Here χ is the heading angle in the horizontal plane and μ is the roll angle. For $\epsilon \to 0$, this reduces to:

$$\dot{E} = [2g(E - h)]^{1/2}(T - D)/W \tag{14}$$

$$\dot{\chi} = (gL/WV) \sin\mu \tag{15}$$

$$0 = L \cos\mu - W \tag{16}$$

This appears interesting, in itself, for use in a coarse approximation to optimal variable-altitude turns. Here the specific energy E and path azimuth angle χ comprise the state, and altitude h and roll angle μ the controls; the angle of attack α being thought of as eliminated by use of Eq. (16).

The Euler equations for this system possess two first integrals, namely

$$H = \lambda_E [V(T - D)/W] + \lambda_\chi(gL/WV) \sin\mu = \text{const} \tag{17}$$

and

$$\lambda_\chi = \text{const} \tag{18}$$

so that extremal arcs can be obtained without numerical integration of the multiplier differential equations. The state system could be solved by numerical integration forward in time, with h and μ obtained by minimization of the pseudo-Hamiltonian function H. By making the further assumption of a parabolic polar, Eq. (19), the optimum roll angle μ may be found analytically.

$$D = D_0(h,V) + L^2(\partial^2 D_i/\partial L^2)(h,V) \tag{19}$$

Substituting Eqs. (16) and (19) into Eq. (17) produces Eq. (20).

$$H = \lambda_E V \frac{T - D_0 - (\partial^2 D_i/\partial L^2)W^2 \sec^2\mu}{W} + \lambda_\chi(g/V) \tan\mu \tag{20}$$

H possesses a single, well-defined maximum with respect to $\tan\mu$ given by Eq. (21).

$$\tan\mu = \frac{\lambda_\chi}{4\lambda_E W(V^2/2g)\, \partial^2 D_i/\partial L^2} \tag{21}$$

With this optimum value of tan μ, the maximization of H with respect to h may proceed as in the symmetric case.

A family of optimal variable-altitude turns, depending on the ratio of initial λ_E to λ_χ, appears in the role of the central paths furnished by the energy curves in the symmetric case. Results obtained from the simplified system should be of considerable interest in providing insight into the gross features of optimal turning maneuvers.

It should be mentioned that the figure traced out in the $(\dot{E},\dot{\chi})$ "hodograph" plane is neither closed nor convex and possesses singularities at $\chi = 90°$ and $V = 0$ for the parabolic polar case. These may be safely ignored only for nearly symmetric flight to increased energy level. For problems involving even momentary energy reduction, e.g. sharp turns, it is appropriate to introduce a second control variable η, $0 \leq \eta \leq 1$, having the character of a throttle/speed brake control, and also a bound $\bar{\alpha}$ on the angle of attack, $|\alpha| \leq \bar{\alpha}$.

$$T = \bar{T}\eta \tag{22}$$

$$D = qS\{C_{D_0} + C_{DB}(1 - \eta) + C_{D\alpha^2}[\alpha^2\eta + \bar{\alpha}^2(1 - \eta)]\} \tag{23}$$

Equation (16) and the assumption of a linear aerodynamic lift curve relate α and μ,

$$\tfrac{1}{2}\rho V^2 S C_{L\alpha}\alpha = W/\cos\mu$$

The bound $\bar{\alpha}$ produces a closed hodograph figure and the term $\bar{\alpha}^2$ $(1-\eta)$ in (23) makes it convex, i.e., it "relaxes" the variational problem,[11] admitting high drag via "chattering" between $\pm\bar{\alpha}$. The analysis of sharp turns will require the maximization of (20) with respect to μ, h, and η, and in the case of a variable-sweep aircraft, with respect to sweep angle Λ also,

Asymptotic Expansion Procedure

The expansion is pursued by writing the Euler equations and expanding all variables in powers of ϵ, a conventional perturbation procedure. The feature requiring special attention is the change in system order at $\epsilon = 0$; hence the problem is termed a *singular perturbation* problem. Corrections describing transitions to and from the zero-order solution and accounting for terms omitted are given as solutions of differential systems *also of reduced order*, which makes this approach attractive computationally.

The introduction of ϵ-dependence into the differential equations so that the variational problem for the reduced system does, in fact, possess a solution, and a solution actually exhibiting appropriate relevance to that of the original system, is a delicate matter and, in the general case, a black art. However, for the present applications, the rationale is provided by the transformation procedure for synthesizing a state differential system canonical form, reported in Refs. 12 and 13, which lays bare the singular extremals in the family.

There is a considerable mathematical literature on singular perturbation problems, surveyed in chapters of two recent books [Wasow (Ref. 14), and Carrier and Pearson (Ref. 15)]. The published research is heavily concentrated on initial value problems, whereas the present interest lies, of course, in two-point boundary value problems. An exploration of singular perturbation procedures for state-Euler differential systems is reported in Ref. 16; however, there is an apparent shortcoming in the treatment given control variables, which is quite different from approximating the results of elimination of the control variables at the outset by maximization of the pseudo-Hamiltonian H.

The assumption that the offending high-order terms in the state system are negligible is tenable neither near the endpoints nor at "corners" of the reduced solution because of the obvious large changes required. Such transitions for the state-Euler system are to be treated in "boundary-layer" approximation (Refs. 14 and 15). A "stretched" time scale is introduced by the transformation $\tau \equiv (t - t_0)/\epsilon$ for solution near the initial point, another, $T \equiv (t_f - t)/\epsilon$ near the terminus, and still others near corners. The boundary layer differential equations approximating transitions to zero-order are obtained by carrying through the transformations and setting $\epsilon \equiv 0$.[14,17] It appears that the zero-order theory may be quite enough for flight performance estimation and, possibly, even for on-board real-time energy management and path control. The procedure for this and higher-order approximations to the solution of initial value problems, due to Vasi'leva, is detailed in Ref. 14. It rests heavily upon Lyapunov stability theory, and there is a central question whether assumed stability properties of the differential system being attacked are really a necessity, or merely a convenience in building the theory. One certainly hopes the latter is the case, because the Euler equations for atmospheric flight of a lifting vehicle are not renowned for stability! The writers regard the stability question, and the two-point boundary value aspects of asymptotic expansion theory, as of main future interest in applying this approach to variational problems.

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Calculation of the Natural Vibrations of a Tapered Swept Back Rudder Fin

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A NOTE[1] recently published in the Journal of Aircraft presented an outline of a direct method for making calculations of generalized masses and stiffnesses of complex structures. Natural frequencies and modes of such structures can be computed by inserting these characteristic quantities into the equations for vibration analysis. An extensive application of this method to a tapered swept back rudder fin of a high-speed aircraft was made. A short survey of this application is presented and results shown can be compared with a relevant ground resonance test. The multicell fin was rigidly clamped at five spar-roots only, L1 to L5 as shown in Fig. 1.

As a first step, a set of assumed modes F_r, satisfying the conditions of the Rayleigh-Ritz method, must be chosen. A suitable Cartesian coordinate system was identified and the components of 51 modes were developed analytically as polynomials with three independent variables. In order to systematize the procedure, the following conceptual approach

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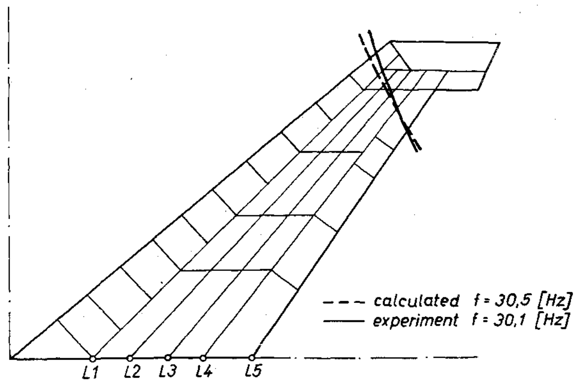

Fig. 1 The 2nd mode.

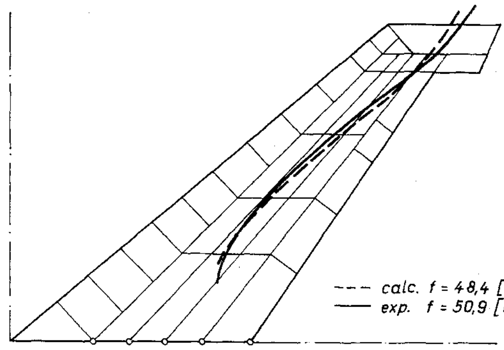

Fig. 2 The 3rd mode.

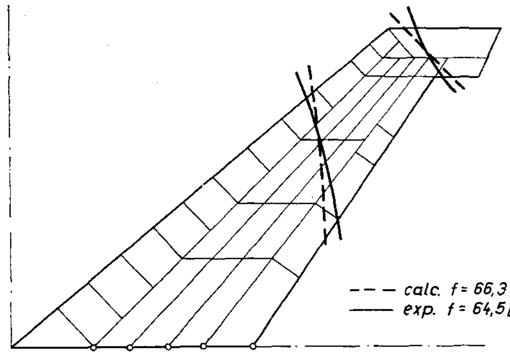

Fig. 3 The 4th mode.

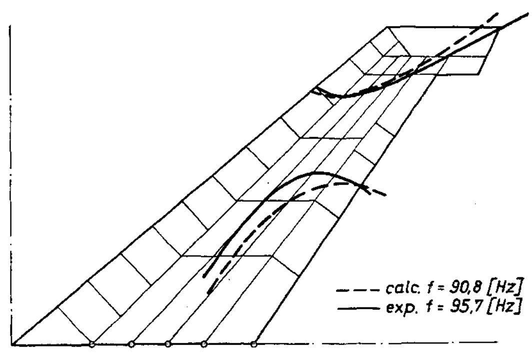

Fig. 4 The 5th mode.